Section 4-3 and 4-4, Mathematics 108

Log Functions

A log function is an inverse exponential function.

Repeat after me,

A log function is an inverse exponential function
A log function is an inverse exponential function
A log function is an inverse exponential function

Log functions are very difficult to understand and work with, unless you remember
A log function is an inverse exponential function

Let's see how this works.

Exponential function

Example:

$$f(x) = 2^x$$

$$f^{-1}(x) = \log_2 x$$

$$f(4) = 2^4 = 16$$

$$f^{-1}(16) = \log_2 16 = 4$$

Example:

$$f(x) = 10^x$$

$$f^{-1}(x) = \log_{10} x$$

$$f(3) = 10^3 = 1000$$

$$f^{-1}(1000) = \log_{10} 1000 = 3$$

Notice:

$f^{-1}(.001) = \log_{10} .001 = -3$ because $10^{-3} = .001$

Example:

$$f(x) = 16^x$$

$$f^{-1}(x) = \log_{16} x$$

$$f\left(\frac{1}{2}\right) = 16^{1/2} = \sqrt{16} = 4$$

$$f^{-1}(4) = \log_{16} 4 = \frac{1}{2}$$

Notice:
$f^{-1}(.001) = \log_{10} .001 = -3$ because $10^{-3} = .001$

Another thing to remember is

A LOG IS AN EXPONENT!

Here's what I mean by that.

$\log_{10} 1000 = 3$

So 3 is an exponent.

That is $10^3 = 1000$

I've been repeating myself here, but that is what you need to do.

You have to do all the homework problems and then do them over and over again.
The concept of exponential functions is easy.
The concept of log functions is hard.
But a log function is just the inverse of an exponential function.

Repeat this over and over again...

Logs are one of the most confusing ideas we will learn.

That doesn't mean you have to be confused.

Look at this over and over and it will sink in.

A Word on Notation

The following is included so to try to de-confuse you on notation.

For an exponential function a^x, the *a* is called the base.

We use the same name for log functions.

So we say $\log_{10} x$ as "Log to the base 10 of *x*"

There are two very common values you will see used as the bases for log functions.

They are 10 and *e* the Euler constant.

Since many of you are future computer scientists I will add that 2 is often used as a log base in computer science. Anyone know why?

There are two interpretations for $\log x$

Among scientists $\log x$ often means $\log_{10} x$
Note this is probably what your calculator means if it has a [LOG] button.

Among mathematician $\log x$ often means $\log_e x$

This is a very confusing situation.

It is made better by the use of

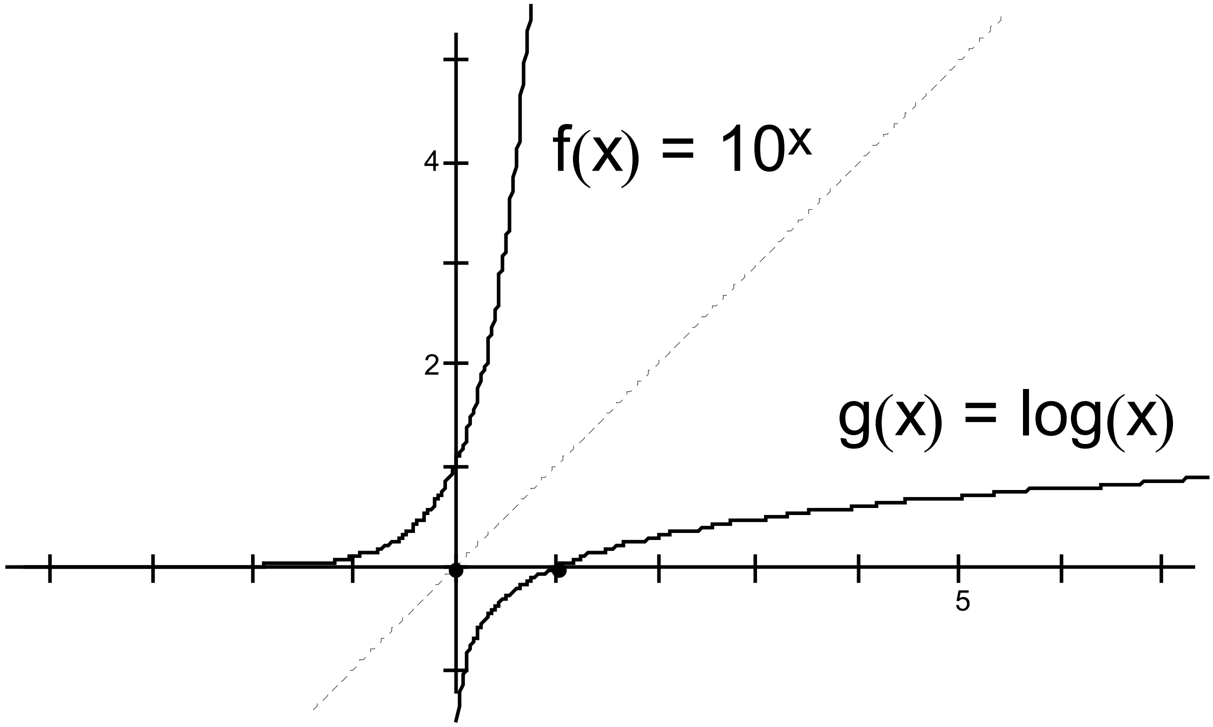
$\ln x = \log_e x$

I will try to never use $\log x$ in this class unless I don't care what the base is.

Instead I will either write $\log_{10} x$ or $\ln x$

Graph of log functions

Since a log function is the inverse of an exponential function, you would expect they would be mirror images across $y=x$ like all inverse functions.



Note that the domain of the exponential function is all reals but its range is $x>0$.

Therefore the log function has a domain of $x>0$ but its range is all reals.

The image of all logs < 0 comes from the domain interval (0,1).

Properties of Log Functions

All of the properties of log functions can be derived from the fact that

Repeat after me,

A log function is an inverse exponential function

1. $a^0 = 1$ $\qquad\qquad$ $\log_a 1 = 0$
(Remember a is the base, 0 is a log so it is an exponent

2. $a^1 = a$ $\qquad\qquad$ $\log_a a = 1$

3. $a^x = a^x$ $\qquad\qquad$ $\log_a a^x = x$

Note that 3. is an expression of the property of inverse functions and the fact that a^x and $\log_a x$ are inverse functions. Number 4. is the other property of inverse functions.

4. $\qquad\qquad$ $a^{\log_a x} = x$

Multiplying and Dividing by adding and multiplying Logs

5. $a^{x+y} = a^x \cdot a^y$ $\qquad\qquad$ $\log_a x + \log_a y = \log_a xy$

6. $a^{x+y} = a^x \cdot a^y$ $\qquad\qquad$ $\log_a x - \log_a y = \log_a \frac{x}{y}$

Stare at these two properties until they make sense.

Consider that

$\left(a^x\right)^2 = a^{2x}$ $\qquad\qquad$ $\log_a x^2 = \log_a x + \log_a x = 2\log_a x$

So we get

7. $\left(a^x\right)^b = a^{bx}$ $\qquad\qquad$ $\log_a x^b = b\log_a x$

Examples:

Calculating

$$\log_2 80 - \log_2 5 = \log_2 \frac{80}{5} = \log_2 16 = \log_2 2^4 = 4:$$

$$\log_4 2 + \log_4 32 = \log_4 32 \cdot 2 = \log_4 64 = \log_4 4^3 = 3$$

$$-\frac{1}{3}\log_{10} 8 = \log_{10} 8^{-1/3} = \log_{10} \frac{1}{8^{1/3}} = \log_{10} \frac{1}{\sqrt[3]{8}} = \log_{10} \frac{1}{2} = -.301$$

The last step requires a table or a calculator.

Expanding

$$\log_2 6x = \log_2 2 + \log_2 3 + \log_2 x = 1 + \log_2 3 + \log_2 x$$

$$\log_5 x^3 y^6 = \log_5 x^3 + \log_5 y^6 = 3\log_5 x + 6\log_5 y$$

$$\ln \frac{ab}{\sqrt[3]{c}} = \ln a + \ln b - \ln \sqrt[3]{c} = \ln a + \ln b - \ln c^{1/3} = \ln a + \ln b - \frac{\ln c}{3}$$

Combining

$$3\log x + \frac{1}{2}\log\left(x+1\right) = \log x^3 + \log \sqrt{x+1} = \log\left(x^3\sqrt{x+1}\right)$$

$$3\ln s + \frac{1}{2}\ln t - 4\ln\left(t^2+1\right) = \ln s^3 + \ln\sqrt{t} - \ln\left(t^2+1\right)^4 = \ln\left[\frac{s^3+\sqrt{t}}{\left(t^2+1\right)^4}\right]$$

Base 10 Logs

History Lesson

For many years before scientific calculators became available and inexpensive, scientists would use base 10 logarithms to do calculations. They would use **Log Tables**.

Log tables would have logs for the numbers $1 < x < 10$. This was called a **Mantissa.** The part of the log that represented a power of 10 was called the **Charactoristic.**

Someone using the tables would need keep track of where the decimal point was.

Here's the process for multiplying two numbers.

1) Look up the first number in the tables and get the mantissa.
This could include doing a linear interpolation between two table values to get one more digit.

2) Add the characteristic.

3) Repeat for the second number

4) Add the two logs together (Remember adding logs is multiplying)

5) look up the new mantissa in the table in reverse, again interpolating to get the 5th digit.

6) Adjust the number's decimal point by the new characteristic.

This was much faster than multiplying two 5 digits numbers, the alternative.

FIVE-PLACE MANTISSAS FOR COMMON LOGARITHMS

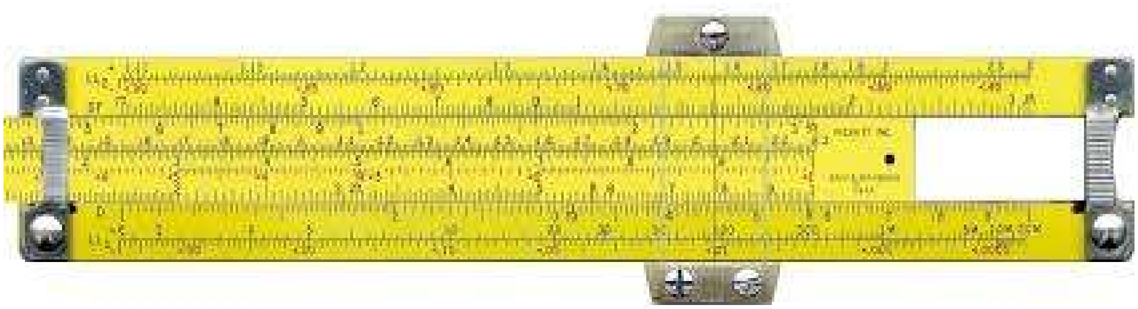
N.	0	1	2	3	4	5	6	7	8	9
100	00 000	043	087	130	173	217	260	303	346	389
101	432	475	518	561	604	647	689	732	775	817
102	860	903	945	988	*030	*072	*115	*157	*199	*242
103	01 284	326	368	410	452	494	536	578	620	662
104	703	745	787	828	870	912	953	995	*036	*078
105	02 119	160	202	243	284	325	366	407	449	490
106	531	572	612	653	694	735	776	816	857	898
107	938	979	*019	*060	*100	*141	*181	*222	*262	*302
108	03 342	383	423	463	503	543	583	623	663	703
109	743	782	822	862	902	941	981	*021	*060	*100
110	04 139	179	218	258	297	336	376	415	454	493
111	532	571	610	650	689	727	766	805	844	883
112	922	961	999	*038	*077	*115	*154	*192	*231	*269
113	05 308	346	385	423	461	500	538	576	614	652
114	690	729	767	805	843	881	918	956	994	*032
115	06 070	108	145	183	221	258	296	333	371	408
116	446	483	521	558	595	633	670	707	744	781
117	819	856	893	930	967	*004	*041	*078	*115	*151
118	07 188	225	262	298	335	372	408	445	482	518
119	555	591	628	664	700	737	773	809	846	882
120	918	954	990	*027	*063	*099	*135	*171	*207	*243
121	08 279	314	350	386	422	458	493	529	565	600
122	636	672	707	743	778	814	849	884	920	955
123	991	*026	*061	*096	*132	*167	*202	*237	*272	*307
124	09 342	377	412	447	482	517	552	587	621	656
125	691	726	760	795	830	864	899	934	968	*003
126	10 037	072	106	140	175	209	243	278	312	346
127	380	415	449	483	517	551	585	619	653	687
128	721	755	789	823	857	890	924	958	992	*025
129	11 059	093	126	160	193	227	261	294	327	361
130	394	428	461	494	528	561	594	628	661	694
131	727	760	793	826	860	893	926	959	992	*024
132	12 057	090	123	156	189	222	254	287	320	352
133	385	418	450	483	516	548	581	613	646	678
134	710	743	775	808	840	872	905	937	969	*001
135	13 033	066	098	130	162	194	226	258	290	322
136	354	386	418	450	481	513	545	577	609	640
137	672	704	735	767	799	830	862	893	925	956
138	988	*019	*051	*082	*114	*145	*176	*208	*239	*270
139	14 301	333	364	395	426	457	489	520	551	582
140	613	644	675	706	737	768	799	829	860	891
141	922	953	983	*014	*045	*076	*106	*137	*168	*198
142	15 229	259	290	320	351	381	412	442	473	503
143	534	564	594	625	655	685	715	746	776	806
144	836	866	897	927	957	987	*017	*047	*077	*107
145	16 137	167	197	227	256	286	316	346	376	406
146	435	465	495	524	554	584	613	643	673	702
147	732	761	791	820	850	879	909	938	967	997
148	17 026	056	085	114	143	173	202	231	260	289
149	319	348	377	406	435	464	493	522	551	580
150	609	638	667	696	725	754	782	811	840	869
N.	0	1	2	3	4	5	6	7	8	9

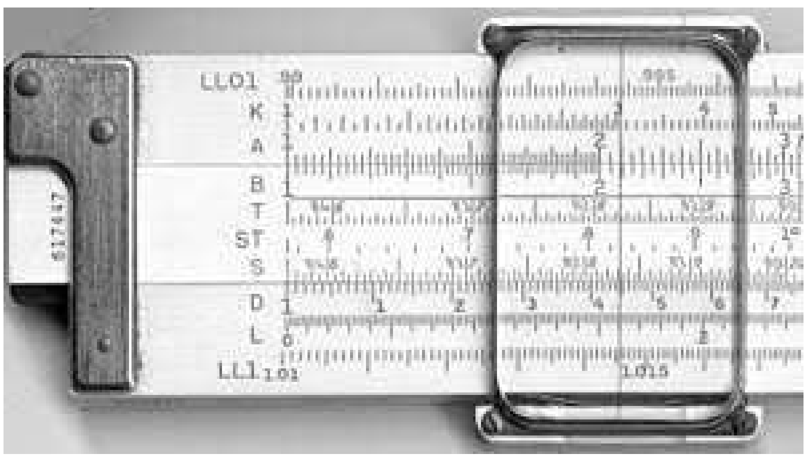
Proportional parts

	44	**43**	**42**
1	4.4	4.3	4.2
2	8.8	8.6	8.4
3	13.2	12.9	12.6
4	17.6	17.2	16.8
5	22.0	21.5	21.0
6	26.4	25.8	25.2
7	30.8	30.1	29.4
8	35.2	34.4	33.6
9	39.6	38.7	37.8
	41	**40**	**39**
1	4.1	4.0	3.9
2	8.2	8.0	7.8
3	12.3	12.0	11.7
4	16.4	16.0	15.6
5	20.5	20.0	19.5
6	24.6	24.0	23.4
7	28.7	28.0	27.3
8	32.8	32.0	31.2
9	36.9	36.0	35.1
	38	**37**	**36**
1	3.8	3.7	3.6
2	7.6	7.4	7.2
3	11.4	11.1	10.8
4	15.2	14.8	14.4
5	19.0	18.5	18.0
6	22.8	22.2	21.6
7	26.6	25.9	25.2
8	30.4	29.6	28.8
9	34.2	33.3	32.4
	35	**34**	**33**
1	3.5	3.4	3.3
2	7.0	6.8	6.6
3	10.5	10.2	9.9
4	14.0	13.6	13.2
5	17.5	17.0	16.5
6	21.0	20.4	19.8
7	24.5	23.8	23.1
8	28.0	27.2	26.4
9	31.5	30.6	29.7
	32	**31**	**30**
1	3.2	3.1	3.0
2	6.4	6.2	6.0
3	9.6	9.3	9.0
4	12.8	12.4	12.0
5	16.0	15.5	15.0
6	19.2	18.6	18.0
7	22.4	21.7	21.0
8	25.6	24.8	24.0
9	28.8	27.9	27.0

Note this is 1 of 20 pages one might use. The last column "Proportional Parts" was used to help with the linear interpolation.

For less accurate but quicker results, scientists and students would use a **Slide Ruler**.





The first ruler is yellow for better contrast since one might have to stare at it a lot.

The different lines on the ruler, eg. A, B, K, D, L are because the ruler could do other calculations such as square and cubed roots.

A slider ruler could cost anywhere from $5 for a cheap plastic one to over a hundred dollars for a fancy aluminum model.

A student carrying a slide ruler around was a sure sign that he was a nerd.

This was when being a nerd was considered a very bad thing.

Natural logs

Natural logs are logs to the base *e*.

The inverse function of the natural log is of course e^x

Why one would use such a strange base and why these are important would be hard to explain before you take calculus.

For now keep in mind that your scientific and/or graphic calculator has a button dedicated to this function, which says something.

The Change of Base Formula

This is the hardest thing to remember with logs.
The formula we will find can be used to change the base of a log.
Start with:

$\log_a x = p$ or

$a^p = x$

taking the log to the base *b* of both sides we get

$\log_b a^p = \log_b x$

or

$p \log_b a = \log_b x$

But $p = \log_a x$

So we get

$\log_a x \cdot \log_b a = \log_b x$

Or finally

$$\log_a x = \frac{\log_b x}{\log_b a}$$

In particular if *x*=b

$$\log_a b = \frac{\log_b b}{\log_b a} = \frac{1}{\log_b a}$$

or $\log_a b \cdot \log_b a = 1$

Example:

Evaluate $\log_8 5$

$$\log_8 5 = \frac{\log_{10} 5}{\log_{10} 8} = .77398$$